REPRESENTATION THEOREMS FOR TCHEBYCHEFFIAN POLYNOMIALS WITH BOUNDARY CONDITIONS AND THEIR APPLICATIONS[†]

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ABSTRACT

Representation theorems for Tchebycheff polynomials with homogeneous boundary conditions are proved, and a number of extremal problems are solved.

Representation theorems for non-negative polynomials were considered by Lukács in the early decades of this century. Lukács proved (see Szegö [13, p. 4]) that every non-trivial polynomial $p_n(t) = \sum_{k=0}^{n} a_k t^k$, non-negative on a finite interval $[a,b]$, admits a representation which is essentially a sum of squares of polynomials. In 1953, Karlin and Shapley [8, p. 35] established for $p_n(t)$ as above, the existence of a unique representation of the form

$$(1)\quad p_n(t) = \begin{cases} \alpha \prod_{j=1}^{m} (t - t_{2j-1})^2 + \beta(t-a)(b-t) \prod_{j=1}^{m-1} (t - t_{2j})^2, & \text{for } n = 2m \\ \alpha(t-a) \prod_{j=1}^{m} (t - t_{2j})^2 + \beta(b-t) \prod_{j=1}^{m} (t - t_{2j-1})^2, & \text{for } n = 2m+1 \end{cases}$$

with $\alpha, \beta > 0$, and $a \leqq t_1 \leqq \cdots \leqq t_{n-1} \leqq b$, where uniqueness is in terms of polynomials with a full set of real zeros. Moreover, for $p_n(t)$ strictly positive on $[a,b]$, the strict inequalities $a < t_1 < t_2 < \cdots < t_{n-1} < b$ hold.

In 1963 and 1966, Karlin further extended these results to polynomials generated by Tchebycheff systems [2], and indicated a host of applications [3].

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In this paper, we generalize the results of Karlin [2], [3], to obtain corresponding representation theorems for Tchebycheffian polynomials satisfying certain general homogeneous boundary conditions. The results are new even in the situation of ordinary polynomials.

Although this paper considers the problem in the setting of ECT-systems (see Section 1), the basic prototype for such systems is $\{t^k\}_0^n$. A reader not totally at ease with the general formulation should interpret all the discussion for the case of the powers.

The organization of the paper is as follows. Section 1 sets forth the basic underlying concepts and terminology. In Section 2, we strive to provide some insight into the nature of the boundary conditions which are fundamental in all that follows. Section 3 highlights the main theorems and records some corollaries, and the final Section 4 exposes a number of applications of the representation theorem in solving certain related extremal problems.

Before entering into the detailed discussion which is technical in nature, it may be useful to indicate a concrete example of one of the main theorems and one of its applications.

Let $f(t)$ be a continuous positive function on $[0,1]$. Consider the class of polynomials of degree n satisfying the boundary conditions

$$\begin{aligned} p^{(j_l)}(0) &= 0, \qquad l = 1,\cdots,k \\ p^{(h_l)}(1) &= 0, \qquad l = 1,\cdots,m \end{aligned} \tag{2}$$

where $0 \leqq j_1 < \cdots < j_k \leqq n$, $0 \leqq h_1 < \cdots < h_m \leqq n$, and the sets $\{j_l\}$, $l = 1,\cdots,k$, $\{h_l\}$, $l = 1,\cdots,m$, satisfy certain meshing conditions to be stipulated later. Assume $n - k - m = 2r \geqq 0$.

Then, there exists a unique polynomial $p^*(t)$ which satisfies (2) and:

(i) $f(t) \geqq p^*(t) \geqq 0$ for $t \in [0,1]$.

(ii) $p^*(t)$ has r distinct zeros, each of multiplicity 2, in $(0,1)$.

(iii) $f(t) - p^*(t)$ vanishes at least once between the adjacent zeros of $p^*(t)$ in $(0,1)$, and at least once between the smallest zero therein and 0, and between the largest zero therein and 1.

The $p^*(t)$ constructed above satisfies many extremal properties. As an example, consider the class of non-negative polynomials which are less than or equal to $f(t)$ on $[0,1]$, and which satisfy (2). Assume $p(1) = 0$ is one of the boundary conditions.

Then, any expression of the form $\sum_{i=0}^{n} a_i(-1)^{\varepsilon_i} p^{(i)}(1)$, where the a_i are non-negative, and the ε_i count the number of $h_l < i$, $l = 1, \cdots, k$, is uniquely maximized among all polynomials in the above class by $p^*(t)$, under certain minor restrictions.

1. Formulation and introduction

Tchebycheff systems are familiar objects of importance in many domains, (for example, they feature prominently in the theory of inequalities, convexity, as successive eigenfunctions in certain differential and integral equations, and in approximation theory) and have been extensively studied. (Consult the treatise of Karlin and Studden [9], for a detailed treatment of the subject.) We record, for ready reference, certain basic facts, to be found in [9].

I. The following notation is convenient. For $t_0 < \cdots < t_p$, let

$$U^*\begin{pmatrix} 0,1,\cdots,p \\ t_0,t_1,\cdots,t_p \end{pmatrix} = U\begin{pmatrix} 0,1,\cdots,p \\ t_0,t_1,\cdots,t_p \end{pmatrix} = \begin{vmatrix} u_0(t_0) & \cdots & u_0(t_p) \\ \vdots & & \vdots \\ u_p(t_0) & \cdots & u_p(t_p) \end{vmatrix} \tag{3}$$

while if $t_{i-1} < t_i = t_{i+1} = \cdots = t_{i+q} < t_{i+q+1}$, then

$$U^*\begin{pmatrix} 0,1,\cdots,p \\ t_0,t_1,\cdots,t_p \end{pmatrix}$$

is the determinant (3), with the $(i+1+j)$th column, $0 \leqq j \leqq q$, replaced by the vector

$$\left(\frac{\partial^j u_0(t_i)}{\partial t^j}, \cdots, \frac{\partial^j u_p(t_i)}{\partial t^j}\right).$$

$\{u_i(t)\}$, $i = 0, \cdots, n$, is defined to be an ECT-system (extended complete Tchebycheff system) on $[a,b]$ if $u_i(t) \in C^n[a,b]$, $i = 0,1,\cdots,n$, and

$$U^*\begin{pmatrix} 0,1,\cdots,p \\ t_0,t_1,\cdots,t_p \end{pmatrix} > 0$$

for all $a \leqq t_0 \leqq \cdots \leqq t_p \leqq b$, and $p = 0,1,\cdots,n$.

II. An equivalent definition, up to the sign of one of the functions, of an ECT-system is the following.

Let $Z_{[a,b]}(f(t))$ denote the number of zeros of $f(t)$ in $[a,b]$, counting multiplicities. Then $\{u_i(t)\}$, $i = 0,\cdots,n$, is an ECT-system on $[a,b]$ if $u_i(t) \in C^n[a,b]$, $i = 0,1,\cdots,n$, and $Z_{[a,b]}(\sum_{i=0}^{p} a_i u_i(t)) \leqq p$, provided $\sum_{i=0}^{p} a_i^2 > 0$, $p = 0,1,\cdots,n$.

III. Pólya pointed out the following characterization of ECT-systems (see Karlin and Studden [9, p. 379]).

THEOREM A. (Pólya.) *Let $u_i(t) \in C^n[a,b]$ obey the initial conditions*

(4) $$u_k^{(p)}(a) = 0 \qquad p = 0,1,\cdots,k-1; \qquad k = 1,\cdots,n.$$

Then the following statements are equivalent:

(i) $\{u_i\}, i = 0,\cdots,n$, *is an* ECT-*system on* $[a,b]$,

(ii) $$u_i(t) = w_0(t)\int_a^t w_1(\xi_1)\cdots\int_a^{\xi_{i-1}} w_i(\xi_i)\, d\xi_i \cdots d\xi_1, \qquad i = 0,1,\cdots,n$$

where $w_0(t),\cdots,w_n(t)$ are $n+1$ strictly positive functions on $[a,b]$ such that $w_k(t)$ is of continuity class $C^{n-k}[a,b]$.

REMARK. If the conditions (4) are not satisfied by an ECT-system $\{u_i\}$, $i = 0,\cdots,n$, then effecting a non-singular linear transformation, we can determine a new ECT-system $\{\tilde{u}_i\}$, $i=0,\cdots,n$, which does satisfy (4), and assume henceforth that this is done. (Note that if $a = 0$ and $w_i(t) = 1/(i+1)$, $i = 0,1,\cdots,n$, then $u_i(t) = t^i_,\ i = 0,1,\cdots n$.)

IV. Associated with an ECT-system $\{u_i\}$, $i = 0,\cdots,n$, is a natural sequence of first order differential operators

$$D_j f = \frac{d}{dt}\frac{f(t)}{w_j(t)}, \qquad j = 0,1,\cdots,n$$

where the $\{w_i(t)\}$, $i = 0,\cdots,n$, are those exhibited in Theorem A. Define

$$D^j = D_{j-1}\cdots D_0, \qquad j = 1,\cdots,n+1, \quad \text{and} \quad D^0 = I.$$

Accordingly,

$$D^j u_j(t) = w_j(t), \qquad j = 0,1,\cdots,n,$$
$$D^k u_j(t) = 0, \qquad \text{for} \quad k > j,$$

and by virtue of the initial condition (4), we have

$$D^k u_j(a) = \delta_{kj} w_j(a), \qquad j = 0,1,\cdots,n.$$

(Note that for the powers $\{t^k\}$, $k = 0,\cdots,n$, $D^j = (1/j!)(d^j/dt^j)$.)

We are now prepared to introduce the boundary conditions to be considered. We shall be interested in polynomials (that is, expressions of the type $u(t) = \sum_{i=0}^{n} a_i u_i(t)$) satisfying homogeneous boundary conditions of the form:

$$(5)\qquad \begin{aligned} \mathscr{A}_k:&\ \sum_{j=0}^{n} A_{ij} D^j u(a) = 0 \qquad i = 1,\cdots,k,\\ \mathscr{B}_m:&\ \sum_{j=0}^{n} B_{ij} D^j u(b) = 0 \qquad i = 1,\cdots,m, \end{aligned}$$

where the matrices $\tilde{A} = \| A_{ij}(-1)^j \|$, $i = 1,\cdots,k$; $j = 0,\cdots,n$, and $B = \| B_{ij} \|$, $i = 1,\cdots,m$; $j = 0,\cdots n$, obey Postulate I below. The collection of all such non-trivial polynomials will be denoted by $\mathscr{U}(\mathscr{A}_k \cap \mathscr{B}_m)$.

POSTULATE I. (i) *$0 \leqq k, m \leqq n$, and $k + m \leqq n$.*

(ii) *There exists $\{i_1,\cdots,i_m\}$, $\{j_1,\cdots.j_k\}$ such that*

$$(6)\qquad \begin{aligned} &B\begin{pmatrix}1,\cdots,m\\ i_1,\cdots,i_m\end{pmatrix} \tilde{A}\begin{pmatrix}1,\cdots,k\\ j_1,\cdots,j_k\end{pmatrix} \neq 0 \ \text{ and}\\ &i_\nu \leqq j'_{\nu+1+n-(m+k)}, \qquad \nu = 1,\cdots,m \end{aligned}$$

where $\{j'_\nu\}$, $\nu = 1,\cdots,n-k+1$, is the complementary set of indices to $\{j_\nu\}$, $\nu = 1,\cdots,k$, in $\{l\}$, $l = 0,\cdots,n$.

(iii) *For all $\{i_1,\cdots,i_m\}$, $\{j_1,\cdots,j_k\}$ satisfying* (ii),

$$\operatorname{sgn} B\begin{pmatrix}1,\cdots,m\\ i_1,\cdots,i_m\end{pmatrix} = \varepsilon_m(B), \text{ and } \operatorname{sgn} \tilde{A}\begin{pmatrix}1,\cdots,k\\ j_1,\cdots,j_k\end{pmatrix} = \varepsilon_k(\tilde{A}),$$

that is, the $m \times m$ and $k \times k$ subdeterminants from (ii) *have constant signs, repectively.*

The possibility that boundary conditions apply only to one endpoint is not excluded. No ambiguity in the terminology should arise.

Two examples of boundary conditions which satisfy Postulate I are as follows.

Example I.

$$\begin{aligned} D^{j_s}u(a) &= 0 \qquad s = 1,\cdots,k\\ D^{i_s}u(b) &= 0 \qquad s = 1,\cdots,m \end{aligned}$$

provided $i_\nu \leqq j'_{\nu+1+n-(m+k)}$, $\nu = 1,\cdots,m$.

Example II. Boundary constraints appearing at one endpoint only, say b, where B has rank m, and all $m \times m$ non-zero subdeterminants are of one sign.

In the later analysis, it will be necessary to work with the concept of degree of a zero at an endpoint in the presence of boundary conditions. This somewhat encumbers the analysis and leads to the following definition and construction.

DEFINITION 1.1. If $u(t)$ in the class $\mathscr{U}(\mathscr{A}_k \cap \mathscr{B}_m)$ entails $u(b) = Du(b) = \cdots = D^{\beta-1}u(b) = 0$, while there exists a $u(t) \in \mathscr{U}(\mathscr{A}_k \cap \mathscr{B}_m)$ for which $D^{\beta}u(b) \neq 0$, then we say that the class of polynomials $\mathscr{U}(\mathscr{A}_k \cap \mathscr{B}_m)$ *has a zero of degree β at b*, denoted by $\beta = N_b(\mathscr{A}_k, \mathscr{B}_m)$. Similarly, we define $N_a(\mathscr{A}_k, \mathscr{B}_m)$.

We emphasize that $N_b(\mathscr{A}_k, \mathscr{B}_m)$, or $N_a(\mathscr{A}_k, \mathscr{B}_m)$, depends on *both* sets of boundary conditions $\mathscr{A}_k$ and $\mathscr{B}_m$. The following simple example illustrates this essential fact.

Let $B = [1,0,0,0,1]$, $\tilde{A} = [0,0,0,0,1]$, $\tilde{C} = [1,0,0,0,0]$. Both B and $\tilde{A}$, and B and $\tilde{C}$ satisfy Postulate I. However, $N_b(\mathscr{A}_1, \mathscr{B}_1) = 1$, while $N_b(\mathscr{C}_1, \mathscr{B}_1) = 0$.

We now outline the procedure used for the *addition of a zero at an endpoint*.

Assume $N_b(\mathscr{A}_k, \mathscr{B}_m) = \beta$, and $m + k < n$. Let $B' = \| B'_{ij} \|$, $i = 1, \cdots, m+1$; $j = 0, \cdots, n$, where

$$B'_{ij} = \begin{cases} \delta_{\beta j} & i = 1; \quad\quad\quad\quad j = 0, 1, \cdots, n \\ B_{i-1,j} & i = 2, \cdots, m+1; \quad j = 0, 1, \cdots, n \end{cases}$$

and let $\mathscr{B}'_{m+1}$: $\sum_{j=0}^{n} B'_{ij} D^j u(b) = 0$, $i = 1, \cdots, m+1$. We call this construction *the addition of a zero at b*. A similar construction may be done at a to obtain $\mathscr{A}'_{k+1}$.

It shall be shown that subject to the stipulations of Postulate I, the extended boundary conditions represented by B' and $\tilde{A}$, and B and $\tilde{A}'$, also satisfy Postulate I.

Postulate I has wide scope in that it is fulfilled for the usual type of boundary conditions occuring in the study of vibrating systems of coupled mechanical systems (see Neumark [11], Gantmacher and Krein [1], and Karlin [4, Chap. 10]). Postulate I may be interpreted as a generalization of the Pólya conditions used in Hermite-Birkhoff interpolation (see [6], [12]). A slightly more restrictive form of Postulate I is usually stated where (iii) prevails when the respective determinants are non-zero, whether $\{i_1, \cdots, i_m\}$, $\{j_1, \cdots, j_k\}$ satisfy (ii) or not (see [5], [7]). Due to the requirement that Postulate I be invariant under the addition of a zero at an endpoint, we find it necessary to introduce the more general definition.

Our principal aim is to generalize the representation theorems of Karlin [2], and their corollaries, to the situation of polynomials satisfying boundary conditions of the above form. The basic representation theorem to be proved is the following.

THEOREM 3.2 (a). *Let* $\{u_i\}$, $i=0,\cdots,n$, *be an* ECT-*system on* $[a,b]$. *Let* $\tilde{A}$ *and* B *be given* $k \times n+1$ *and* $m \times n+1$ *matrices, respectively, satisfying Postulate* I, *and assume* $N_b(\mathscr{A}_k,\mathscr{B}_m) = \beta$ *and* $N_a(\mathscr{A}_k,\mathscr{B}_m) = \alpha$. *If* $f(t)$ *is a continuous function on* $[a,b]$, *positive on* (a,b), *with zeros of degree* γ *and* δ *at* a *and* b *respectively, where* $\gamma \leqq \alpha$, $\delta \leqq \beta$, *then, for* $n-m-k=2r \geqq 0$, *there exists a polynomial* $u^*(t)$ *for which*

(i) $f(t) \geqq u^*(t) \geqq 0$, *for all* $t \in [a,b]$,

(ii) $u^* \in \mathscr{U}(\mathscr{A}_k \cap \mathscr{B}_m)$,

(iii) $Z_{(a,b)}(u^*(t)) = 2r$, *with* r *distinct zeros,*

(iv) $f(t) - u^*(t)$ *vanishes at least once between each pair of adjacent interior zeros of* $u^*(t)$, *and at least once between the largest zero and the endpoint* b, *and between the smallest zero and the endpoint* a.

The uniqueness of $u^*(t)$ is not guaranteed, as in Karlin [2], without further stipulations. This is due to the non-independence of the addition of a zero at a, and the addition of a zero at b. To deal with this problem we define Property J.

Property J. *We say that* $\mathscr{U}(\mathscr{A}_k \cap \mathscr{B}_m)$ *satisfies Property* J *if* $m+k<n$, *and the following hold*:

(i) $N_a(\mathscr{A}_k,\mathscr{B}_m) = N_a(\mathscr{A}_k,\mathscr{B}'_{m+1})$ *and*

(ii) $N_b(\mathscr{A}_k,\mathscr{B}_m) = N_b(\mathscr{A}'_{k+1},\mathscr{B}_m)$.

It will be proved that (i) implies (ii) and conversely. Examples underscoring the need for Property J are given in Section 2. Note that Examples I and II given earlier satisfy Property J.

With this definition, we may now state Theorem 3.2 (b).

THEOREM 3.2 (b). *The* $u^*(t)$ *in Theorem* 3.2 (a) *is unique if* one *of the following holds.*

(i) $(\alpha+\beta) > (\gamma+\delta)$,

(ii) $\alpha+\beta = \gamma+\delta$, *but* $f^{(\alpha)}(a) \neq u^{*(\alpha)}(a)$ *or* $f^{(\beta)}(b) \neq u^{*(\beta)}(b)$,

(iii) *Property* J *holds for* $\mathscr{U}(\mathscr{A}_k \cap \mathscr{B}_m)$.

As consequences of the representation theorem, we establish representations for $u(t) \in \mathscr{U}(\mathscr{A}_k \cap \mathscr{B}_m)$, $u(t) \geqq 0$ for all $t \in [a,b]$, in the spirit of (1). Since the problem divides into various sections, it is not stated here, but is the content of Corollaries 3.2 and 3.3.

The following is an example of one of the applications of interest stemming from the representation theorems.

Consider the collection $\{C_j\}$, $j=0,\cdots,n$, of real numbers and the class of polynomials $u(t) \in \mathscr{U}(\mathscr{A}_k \cap \mathscr{B}_m)$ which lie between 0 and $f(t)$, for all $t \in [a, b]$. We wish to determine the function $u(t)$ in the above class which maximizes or minimizes $\sum_{j=0}^{n} C_j D^j u(b)$. In many cases the polynomial $u^*(t)$ featured in Theorem 3.2 is an extremal polynomial. This is the content of Theorems 4.1 and 4.2.

There is a natural relationship between representation theorems, quadrature formulae, and approximation theory, which is not set forth in this work. By further pursuing the methods of this paper, many of the results of Micchelli-Rivlin [10] on quadrature formulae, which partly provided the motivation for this work, are extendable to situations involving more general boundary conditions of the type (5) satisfying Postulate I.

The important application of representation theorems to problems of best approximation is considered in a forthcoming paper.

2. Preliminaries

The fundamental underlying result of this paper is a variation of Karlin [5, Th. 2]. The theorem decisively rests on the fact that $\{u_i\}$, $i = 0, \cdots, n$, is an ECT-system and on Postulate I.

THEOREM 2.1. *Let* $\{t_i\}$, $i = 1, \cdots, n - (m + k) + 1$, *be any* $n - (m + k) + 1$ *points in* (a, b), $a < t_1 < \cdots < t_{n-(m+k)+1} < b$ *and let* $\{c_i\}$, $i = 1, \cdots, n - (m + k) + 1$, *be given real numbers. Then there exists a unique* $u \in \mathscr{U}(\mathscr{A}_k \cap \mathscr{B}_m)$ *such that* $u(t_i) = c_i$, $i = 1, \cdots, n - (m + k) + 1$.

PROOF. The proof may be found, with minor alterations, in Karlin [5]. Recall that Postulate I as stipulated in this paper is slightly more general.

REMARK 2.1. It is not necessary that the t_i be distinct. Coincidences are allowed, where if $t_{i-1} < t_i = \cdots = t_{i+q} < t_{i+q+1}$ then we ask that $u(t)$ satisfy $u(t_i) = c_i$, $u'(t_i) = c_{i+1}, \cdots, u^{(q)}(t_i) = c_{i+q}$. However, it is essential that the inequalities $a < t_1$, $t_{n-(m+k)+1} < b$ be maintained.

REMARK 2.2. Given $u \in \mathscr{U}(\mathscr{A}_k \cap \mathscr{B}_m)$ such that

$$u(t_i) = 0, \ i = 1, \cdots, n - (m + k), \tag{7}$$

where $a < t_1 \leqq \cdots \leqq t_{n-(m+k)} < b$, (subject to the concept of multiplicities of zeros in the case of equalities of the t_i, as outlined in Remark 2.1), then by Theorem 2.1, there exists a non-trivial $u \in \mathscr{U}(\mathscr{A}_k \cap \mathscr{B}_m)$ satisfying Theorem 2.1, which is unique up to a multiplicative constant. Thus, we may regard $\mathscr{U}(\mathscr{A}_k \cap \mathscr{B}_m)$ as having a basis of $n - (m + k)$ functions which are a Tchebycheff-system on (a, b).

We now present, without proof, a sequence of useful propositions and lemmas. Our first aim is to show that Postulate I holds after an addition of a zero at an endpoint.

LEMMA 2.1. *Let $\tilde{A}$ and B satisfy Postulate* I, *and let* $N_a(\mathscr{A}_k,\mathscr{B}_m) = \alpha$ *and* $N_b(\mathscr{A}_k,\mathscr{B}_m) = \beta$. *Then for all selections* $\{i_1,\cdots,i_m\}$, $\{j_1,\cdots,j_k\}$ *satisfying*

$$B\begin{pmatrix}1,\cdots,m\\ i_1,\cdots,i_m\end{pmatrix}\tilde{A}\begin{pmatrix}1,\cdots,k\\ j_1,\cdots,j_k\end{pmatrix}\neq 0, \text{ and} \tag{8}$$

$$i_v \leqq j'_{v+1+n-(m+k)}, \qquad v = 1,\cdots,m$$

(*that is, obeying condition* (ii) *of Postulate* I), *we have*

$$i_1 = 0,\cdots,i_\beta = \beta - 1 \text{ and} \tag{9}$$

$$j_1 = 0,\cdots,j_\alpha = \alpha - 1. \tag{10}$$

REMARK 2.3. It is readily checked that $i_v \leqq j'_{v+1+n-(m+k)}$, $v = 1,\cdots,m$, and $j_\mu \leqq i'_{\mu+1+n-(m+k)}$, $\mu = 1,\cdots,k$, are equivalent statements.

LEMMA 2.2. *Let* $\mathscr{U}(\mathscr{A}_k\cap\mathscr{B}_m)$ *have zeros of degree* α *and* β *at* a *and* b *respectively. Then*

(i) *there exists* $\{i_1,\cdots,i_m\}$, $\{j_1,\cdots,j_k\}$ *satisfying* (8) *for which* $i_p\neq\beta$, $p = 1,\cdots,m$;

(ii) *there exists* $\{i_1,\cdots,i_m\}$, $\{j_1,\cdots,j_k\}$ *satisfying* (8) *for which* $j_p\neq\alpha$, $p = 1,\cdots,k$.

REMARK 2.4. It is not necessarily true that (i) and (ii) of Lemma 2.2 can be simultaneously satisfied. Indeed consider

$$\tilde{A} = \begin{bmatrix}1&0&0&0&0&-1&0\\0&0&0&0&1&0&0\end{bmatrix},\quad B = \begin{bmatrix}1&0&0&0&0&-1&0\\0&0&0&0&1&0&0\end{bmatrix}.$$

Then $\mathscr{A}_2$ and $\mathscr{B}_2$ satisfy Postulate I while $N_a(\mathscr{A}_2,\mathscr{B}_2) = N_b(\mathscr{A}_2,\mathscr{B}_2) = 0$. If (i) and (ii) were to hold simultaneously, then it would be necessary that $\{4,5\}$, $\{4,5\}$ satisfy condition (ii) of Postulate I. However, the inequality $i_v\leqq j'_{v+1+n-(m+k)}$, $(n = 6,\ m = 2,\ k = 2)$ does not hold for $v = 1$.

PROPOSITION 2.1. *Let $\tilde{A}$ and B satisfy Postulate* I, *and* $m + k < n$. *Let* B' *and* $\mathscr{B}'_{m+1}$ *be obtained from* B *and* $\mathscr{B}_m$ *by the addition of a zero at* b. *Then* $\tilde{A}$ *and* B' *satisfy Postulate* I. (*Similarly for* $\tilde{A}'$ *and* B.)

Corollary 2.1. *Assume that* $\beta = N_b(\mathscr{A}_k, \mathscr{B}_m)$. *Then for all* $u \in \mathscr{U}(\mathscr{A}_k \cap \mathscr{B}_m)$ *with* $n - (m + k)$ *zeros in* (a, b), $D^{\beta}u(b) \neq 0$.

In the consideration of Property J, the following proposition is basic.

Proposition 2.2. *In the definition of Property* J, *statements* (i) *and* (ii) *are equivalent*, (*that is, for* $m + k < n$,

$$N_a(\mathscr{A}_k, \mathscr{B}_m) = N_a(\mathscr{A}_k, \mathscr{B}'_{m+1}) \Leftrightarrow N_b(\mathscr{A}_k, \mathscr{B}_m) = N_b(\mathscr{A}'_{k+1}, \mathscr{B}_m)).$$

Remark 2.5. Two observations result from Proposition 2.2.

(i) Property J does *not* hold if and only if

$$\mathscr{U}(\mathscr{A}_k \cap \mathscr{B}'_{m+1}) = \mathscr{U}(\mathscr{A}'_{k+1} \cap \mathscr{B}_m).$$

(ii) If Property J holds, then $\mathscr{U}(\mathscr{A}'_{k+1} \cap \mathscr{B}''_{m+1}) = \mathscr{U}(\mathscr{A}''_{k+1} \cap \mathscr{B}'_{m+1})$ for $m + k \leqq n - 2$, where $\mathscr{U}(\mathscr{A}'_{k+1} \cap \mathscr{B}''_{m+1})$ denotes the addition of a zero at a, followed by the addition of a zero at b, and analogously for $\mathscr{U}(\mathscr{A}''_{k+1} \cap \mathscr{B}'_{m+1})$.

As a consequence of Property J, the following property holds.

Proposition 2.3. *Assume* $\mathscr{U}(\mathscr{A}_k \cap \mathscr{B}_m)$ *does not satisfy Property* J. *Then for all* $u(t) = \sum_{i=0}^{n} a_i u_i(t) \in \mathscr{U}(\mathscr{A}'_{k+1} \cap \mathscr{B}_m) = \mathscr{U}(\mathscr{A}_k \cap \mathscr{B}'_{m+1})$, $a_n \equiv 0$.

Proposition 2.3 invites questions pertaining to relaxations of Property J. To dispell any such inclinations, the following examples are given.

In each of the examples, $\tilde{A}$ and B satisfy Postulate I.

Example I. $\tilde{A}$ and B satisfy Property J and $a_n = 0$.

$$\tilde{A} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Example II. $\tilde{A}$ and B satisfy Property J and $a_n \neq 0$.

$$\tilde{A} = [1 \quad 0 \quad 0 \quad 0] \qquad B = [1 \quad 0 \quad 0 \quad 0]$$

Example III. $\tilde{A}$ and B do *not* satisfy Property J and $a_n = 0$.

$$\tilde{A} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad B = [1 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0]$$

Example IV. $\tilde{A}$ and B do *not* satisfy Property J and $a_n \neq 0$.

$$\tilde{A} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

3. Representation theorems

Section 2 has laid the groundwork for the representation theorems which follow. In order to prove the main Theorem 3.2, we need the following, which is also of independent interest.

THEOREM 3.1. *Let $f(t)$ be a continuous and strictly positive function on $[c, d]$. Let $\{u_i\}$, $i = 0, \cdots, n$, be an* ECT-*system on $[a, b]$, where $a < c < d < b$. Let $\tilde{A}$ and B be given $k \times n + 1$ and $m \times n + 1$ matrices respectively, satisfying Postulate* I. *Then if p and q are non-negative integers for which $n - m - k - p - q = 2r \geqq 0$, there exists a unique polynomial $u^*_{p,q}(t)$ satisfying the following statements.*

(i) *$f(t) \geqq u(t) \geqq 0$, for all $t \in [c, d]$.*

(ii) *$u \in \mathscr{U}(\mathscr{A}_k \cap \mathscr{B}_m)$.*

(iii) *$Z_{(c,d)}(u(t)) = 2r$, with r distinct zeros.*

(iv) *$u(x_1) = \cdots = u(x_p) = 0$, where $a < x_1 \leqq \cdots \leqq x_p \leqq c$, and $u(y_1) = \cdots = u(y_q) = 0$, where $d \leqq y_1 \leqq \cdots \leqq y_q < b$, with the usual definition of multiplicities in the case of equal x_i and/or y_j.*

(v) *$f(t) - u(t)$ vanishes at least once between each pair of adjacent zeros of $u(t)$ in (c, d), and at least once between the largest zero and the endpoint d, and at least once between the smallest zero and the endpoint c.*

PROOF. Assume without loss of generality, that $x_p < c < d < y_1$. Since $\mathscr{U}(\mathscr{A}_k \cap \mathscr{B}_m)$ has a basis of $n - k - m - p - q + 1$ functions which constitute a Tchebycheff-system on $[c, d]$, the theorem basically follows by the methods of Karlin [2]. Q.E.D.

THEOREM 3.2 (a). *Let $\{u_i\}$, $i = 0, \cdots, n$, be an* ECT-*system on $[a, b]$. Let $\tilde{A}$ and B be given $k \times n + 1$ and $m \times n + 1$ matrices respectively, satisfying Postulate* I, *and assume $N_b(\mathscr{A}_k, \mathscr{B}_m) = \beta$ and $N_a(\mathscr{A}_k, \mathscr{B}_m) = \alpha$. If $f(t)$ is a continuous function in $[a, b]$ positive on (a, b), with zeros of degree γ and δ at a and b respectively, where $\gamma \leqq \alpha$ and $\delta \leqq \beta$, then for $n - m - k = 2r \geqq 0$, there exists a polynomial $u^*(t)$ satisfying the conditions*

(i) *$f(t) \geqq u(t) \geqq 0$, for all $t \in [a, b]$,*

(ii) *$u \in \mathscr{U}(\mathscr{A}_k \cap \mathscr{B}_m)$,*

(iii) *$Z_{(a,b)}(u(t)) = 2r$, with r distinct zeros.*

(iv) *$f(t) - u(t)$ vanishes at least once between each pair of adjacent interior zeros of $u(t)$ and at least once between the largest zero and the endpoint b, and between the smallest zero and the endpoint a.*

REMARK 3.1. The theorem has been stated for $n-m-k = 2r \geqq 0$. If $n-m-k = 2r+1 \geqq 1$, then the same theorem holds, by Proposition 2.1, on replacing $\tilde{A}$ and B by $\tilde{A}'$ and B, or by $\tilde{A}$ and B'.

PROOF OF THEOREM 3.2 (a). For $r = 0$, the proof is simple. Assume $r > 0$. By Theorem 3.1, for each $\varepsilon_1, \varepsilon_2 > 0$ and sufficiently small, that is, $a + \varepsilon_1 < b - \varepsilon_2$, there exists a unique polynomial $u(\varepsilon_1, \varepsilon_2; t)$ satisfying the conditions of Theorem 3.1, where $p = q = 0$, and $[c, d]$ is replaced by $[a + \varepsilon_1, b - \varepsilon_2]$.

Let $\{t_i(\varepsilon_1, \varepsilon_2)\}$, $i = 1, \cdots, r$, be the distinct zeros of $u(\varepsilon_1, \varepsilon_2; t)$ in $(a + \varepsilon_1, b - \varepsilon_2)$, where

$$a + \varepsilon_1 < t_1(\varepsilon_1, \varepsilon_2) < \cdots < t_r(\varepsilon_1, \varepsilon_2) < b - \varepsilon_2,$$

and let $\{s_i(\varepsilon_1, \varepsilon_2)\}$, $i = 1, \cdots, r+1$, be distinct zeros of $f(t) - u(\varepsilon_1, \varepsilon_2; t)$ in $[a + \varepsilon_1, b - \varepsilon_2]$, where

$$a + \varepsilon_1 \leqq s_1(\varepsilon_1, \varepsilon_2) < t_1(\varepsilon_1, \varepsilon_2) < \cdots < t_r(\varepsilon_1, \varepsilon_2) < s_{r+1}(\varepsilon_1, \varepsilon_2) \leqq b - \varepsilon_2.$$

Let $u(\varepsilon_1, \varepsilon_2; t) = \sum_{i=0}^n a_i(\varepsilon_1, \varepsilon_2)\, u_i(t)$, and

$$v(\varepsilon_1, \varepsilon_2; t) = \sum_{i=0}^n a_i(\varepsilon_1, \varepsilon_2) u_i(t) \left[\sum_{i=0}^n [a_i(\varepsilon_1, \varepsilon_2)]^2 \right]^{-\frac{1}{2}},$$

where for ease of notation, we denote

$$\delta^{\varepsilon_1, \varepsilon_2} = \left[\sum_{i=0}^n [a_i(\varepsilon_1, \varepsilon_2)]^2 \right]^{\frac{1}{2}}.$$

By counting zeros, it follows that for fixed ε_2, $t_i(\varepsilon_1, \varepsilon_2)\downarrow$ and $s_j(\varepsilon_1, \varepsilon_2)\downarrow$ as $\varepsilon_1\downarrow$, $i = 1, \cdots, r$; $j = 1, \cdots, r+1$. By the uniqueness of $v(\varepsilon_1, \varepsilon_2; t)$, and since $v(\varepsilon_1, \varepsilon_2; t)$ is uniformly bounded, $v(\varepsilon_1, \varepsilon_2; t)$ is a continuous function of ε_1 (for ε_2 fixed) and analogously, a continuous function of ε_2 (for ε_1 fixed).

In what follows, fix ε_2, and suppress it in the notation. Accordingly,

$$v(\varepsilon_1; t) = \sum_{i=0}^n b_i(\varepsilon_1)\, u_i(t), \text{ where } \sum_{i=0}^n [b_i(\varepsilon_1)]^2 = 1.$$

Thus, as $\varepsilon_1 \downarrow 0$, there exists a subsequence $\{\varepsilon_1^k\}$, $k = 1, \cdots, \infty$, such that $\varepsilon_1^k \downarrow 0$, and $b_i(\varepsilon_1^k) \to b_i$, as $k \to \infty$, $i = 0, \cdots, n$, while $t_i(\varepsilon_1^k) \downarrow t_i$ as $k \to \infty$, $i = 1, \cdots, r$, $s_j(\varepsilon_1^k) \downarrow s_j$ as $k \to \infty$, $j = 1, \cdots, r+1$. Hence, $v(\varepsilon_1^k; t) \to_{k\to\infty} v(t) = \sum_{i=0}^n b_i u_i(t)$ uniformly on $[a, b]$, and $\sum_{i=1}^n b_i^2 = 1$. Since $f(t) \geqq \delta^{\varepsilon_1^k} v(\varepsilon_1; t)$, for all $t \in [a + \varepsilon_1^k, b - \varepsilon_2]$, and $v(\varepsilon_1^k, t) \to_{k\to\infty} v(t) \not\equiv 0$, it follows that $\infty > C \geqq \delta^{\varepsilon_1^k} \geqq 0$, $k = 1, 2, \cdots$, $a \leqq s_1 \leqq t_1 \leqq \cdots \leqq t_r \leqq s_{r+1} \leqq b - \varepsilon_2$, and $\delta^{\varepsilon_1^k} \to_{k\to\infty} \delta$, where $0 \leqq \delta < \infty$.

We show that $t_1 > a$, and $\delta > 0$. Assume $t_1 = a$. Thus $s_1 = a$ as well. $f(t)$ has a zero of degree $\gamma \leqq \alpha$ at a, and $u(\varepsilon_1^k; t)$ has a zero of degree α at a, for all k. Therefore $f(t) - u(\varepsilon_1^k; t)$ has a zero of degree σ at a, where $\sigma \geqq \gamma$. Since $s_1(\varepsilon_1^k)\downarrow a$, $f(s_1(\varepsilon_1^k)) - u(\varepsilon_1^k; s(\varepsilon_1^k)) = 0$, there exists $x(\varepsilon_1^k) \in [a, s_1(\varepsilon_1^k)]$ for which $(f - u(\varepsilon_1^k))^{(i)}\ (x(\varepsilon_1^k)) = 0$, $i = 0, \cdots, \sigma$. Because $t_1(\varepsilon_1^k) \downarrow a$ and $u(\varepsilon_1^k; t_1(\varepsilon_1^k)) = u'(\varepsilon_1^k; t_1(\varepsilon_1^k)) = 0$ there exists $y(\varepsilon_1^k) \in [a, t_1(\varepsilon_1^k)]$ for which $u^{(i)}(\varepsilon_1^k; y(\varepsilon_1^k)) = 0$, $i = 0, \cdots, \alpha + 1$.

$$u(\varepsilon_1^k; t) = \delta^{\varepsilon_1^k} v(\varepsilon_1^k; t) \underset{k \to \infty}{\to} \delta v(t) = u(t)$$

uniformly and continuously on $[a, b]$, and $0 \leqq u(t) \leqq \tilde{C}$, for all $t \in [a, b]$. Thus, $u(t)$ has a zero of degree at least $\alpha + 2$ at a, (note that it may be that $u \equiv 0$), and $(f - u)\ (t)$ has a zero of degree at least $\sigma + 1$ at a. Since $\sigma + 1, \alpha + 2 \geqq \gamma + 1$, $f(t)$ has a zero of degree at least $\gamma + 1$ at a, contradicting the definition of γ. Thus, $t_1 > a$.

Now, $b - \varepsilon_2 \geqq s_2(\varepsilon_1^k) \downarrow s_2 \geqq t_1 > a$. Thus,

$$0 < m = \min_{t \in [s_2, s_2 + \eta]} f(t) \leqq \delta^{\varepsilon_1^k} \max_{t \in [s_2, s_2 + \eta]} v(\varepsilon_1^k; t), \text{ for } k \geqq K_\eta, \text{ and}$$

$$\max_{t \in [s_2, s_2 + \eta]} v(\varepsilon_1^k; t) \leqq \max_{t \in [s_2, s_2 + \eta]} \sum_{i=0}^{n} \left| b_i(\varepsilon_1^k) \right| u_i(t)$$

$$\leqq \max_{t \in [s_2, s_2 + \eta]} \sum_{i=0}^{n} u_i(t) = \tilde{M}.$$

Therefore $\delta^{\varepsilon_1^k} \geqq m / M > 0$, $0 < \delta < \infty$, and $u(t) = \delta v(t) \neq 0$.

Since $u(t_i) = 0$, $i = 1, \cdots, r$, $u(s_i) = f(s_i)$, $i = 1, \cdots, r + 1$, and $t_1 > a$, it follows that

$$a \leqq s_1 < t_1 < \cdots < t_r < s_{r+1} \leqq b - \varepsilon_2.$$

Reintroduce the suppressed index ε_2, that is, $u(t) = u(\varepsilon_2; t)$. We have shown that $u(\varepsilon_2; t)$ is a polynomial satisfying the conditions of Theorem 3.1, with $p = q = 0$ and $[a, b - \varepsilon_2]$ in place of $[c, d]$.

Let $u_1(t)$, $u_2(t)$ satisfy the above. Then $u_1(t) - u_2(t) \in \mathscr{U}(\mathscr{A}_k \cap \mathscr{B}_m)$ and $u_1 - u_2$ has at least $2r$ zeros in $(a, b - \varepsilon_2]$. If $u_1 - u_2$ has more than $2r$ zeros in (a, b) $u_1 \equiv u_2$ by Theorem 2.1. If, on the other hand, $(u_1 - u_2)\ (t)$ has exactly $2r$ zeros in $(a, b - \varepsilon_2]$, then $u_1^{(\alpha)}(a) = u_2^{(\alpha)}(a)$, and $u_1 - u_2 \in \mathscr{U}(\mathscr{A}'_{k+1} \cap \mathscr{B}_m)$. Thus, $u_1 \equiv u_2$, and $u(\varepsilon_2; t)$ is unique for $\varepsilon_2 > 0$.

Analogous to the situation considered, we let $\varepsilon_2 \downarrow 0$, and obtain $u^*(t)$ which satisfies conditions (i)–(iv) of the theorem. Q.E.D.

Theorem 3.2 (b) examines conditions for uniqueness.

THEOREM 3.2 (b). *The $u^*(t)$ in Theorem 3.2 (a) is unique if one of the following holds:*

(i) $(\alpha + \beta) > (\gamma + \delta)$.

(ii) $\alpha + \beta = \gamma + \delta$, *but* $f^{(\alpha)}(a) \neq u^{*(\alpha)}(a)$ *or* $f^{(\beta)}(b) \neq u^{*(\beta)}(b)$.

(iii) *Property* J *holds for* $\mathscr{U}(\mathscr{A}_k \cap \mathscr{B}_m)$.

REMARK 3.2. The $u^*(t)$ mentioned in Remark 3.1 are unique by part (i) of Theorem 3.2 (b). For $n-m-k = 2r+1 \geqq 1$, as in Remark 3.1, denote by $\underline{u}(t)$ the unique polynomial satisfying (i), (iii), (iv) and

(ii′) $u \in \mathscr{U}(\mathscr{A}'_{k+1} \cap \mathscr{B}_m)$;

denote by $\bar{u}(t)$ the unique polynomial satisfying (i), (iii), (iv), and

(ii″) $u \in \mathscr{U}(\mathscr{A}_k \cap \mathscr{B}'_{m+1})$.

If Property J does not hold, then $\bar{u}(t) \equiv \underline{u}(t)$.

PROOF OF THEOREM 3.2 (b). If $u_1(t)$, $u_2(t)$ both satisfy (i)–(iv), then $u_1 - u_2 \in \mathscr{U}(\mathscr{A}_k \cap \mathscr{B}_m)$, and $(u_1 - u_2)(t)$ has at least $2r-1$ zeros in (a,b). It has exactly $2r-1$ zeros in (a,b) if and only if

$$\gamma = \alpha \text{ and } u_1^{(\alpha)}(a) = u_2^{(\alpha)}(a) = f^{(\alpha)}(a) \tag{11}$$

and

$$\delta = \beta \text{ and } u_1^{(\beta)}(b) = u_2^{(\beta)}(b) = f^{(\beta)}(b). \tag{12}$$

If it has $2r$ zeros in (a,b), then either (11) or (12) must hold. Since $(u_1 - u_2)(t) \not\equiv 0$ can have at most $2r$ zeros in (a,b) by Theorem 2.1, we need only consider the above two cases.

Assume $(u_1 - u_2)(t)$ has $2r$ zeros in (a,b) and assume, without loss of generality, that (i) holds. Thus, $u_1 - u_2 \in \mathscr{U}(\mathscr{A}'_{k+1} \cap \mathscr{B}_m)$. But $2r + k + 1 + m = n + 1$ and hence $u_1 \equiv u_2$. By (11) and (12), if (i) or (ii) holds for u_1 or u_2, then $(u_1 - u_2)(t)$ cannot have $2r-1$ zeros in (a,b) and we are finished.

Assume $(u_1 - u_2)(t)$ has $2r-1$ zeros in (a,b) and Property J holds for $\mathscr{U}(\mathscr{A}_k \cap \mathscr{B}_m)$. Then $(u_1 - u_2)^{(\alpha)}(a) = (u_1 - u_2)^{(\beta)}(b) = 0$, and we may add a zero at a and a zero at b to obtain $\mathscr{U}(\mathscr{A}'_{k+1} \cap \mathscr{B}''_{m+1})$ which contains $u_1 - u_2$. Again we have $(2r-1) + (m+1) + (k+1) = n+1$ conditions, and $u_1 \equiv u_2$. Hence the theorem is proved. Q.E.D.

REMARK 3.3. If $N_a(\mathscr{A}_k, \mathscr{B}_m)=N_b(\mathscr{A}, \mathscr{B}_m) = 0$, and Property J holds, then $\mathscr{U}(\mathscr{A}_k \cap \mathscr{B}_m)$ has a basis which is a Tchebycheff-system on $[a, b]$.

Let us return to a consideration of the situation in Theorem 3.1, and for sake of convenience alone, consider the case where $x_p < c < d < y_1$.

COROLLARY 3.1. *Let $f(t)$ in Theorem 3.1 be a polynomial $u(t) \in \mathscr{U}(\mathscr{A}_k \cap \mathscr{B}_m)$ satisfying condition* (iv). *Let $n = m + k + p + q + 2r$, where $r > 0$. Let $u^*_{p,q}(t)$ be as in Theorem* 3.1, *and let $u^{**}_{p,q}(t)$ be the unique polynomial satisfying the conditions* (i), (ii), (v), (iii) *with $2r$ replaced by $2(r-1)$, and*

(iv′) *conditions* (iv) *and $u(c) = u(d) = 0$.*

Then,

$f(t) = u(t) = u^{**}_{p,q}(t) + u^*_{p,q}(t)$ *where the zeros of $u^{**}_{p,q}(t)$ strictly interlace the zeros of $u^*_{p,q}(t)$ in (c, d).*

PROOF. $(u - u^*_{p,q})(t)$ satisfies (i), (ii) and (iv). Moreover, $(u - u^*_{p,q})(t)$ can have at most $2r$ zeros in $[c, d]$ and since $u^*_{p,q}(t)$ satisfies (v), we have

$$(u - u^*_{p,q})(c) = (u - u^*_{p,q})(d) = 0,$$

and $(u - u^*_{p,q})(t)$ has $r-1$ distinct zeros each of multiplicity two in (c, d), interlacing the zeros of $u^*_{p,q}(t)$. By the uniqueness of $u^{**}_{p,q}(t)$, $u(t) - u^{**}_{p,q}(t) = u_{p,q}(t)$.
Q.E.D.

In general, associated with each $\underline{u}(t)$ or $u^*(t)$ (as in Remark 3.2 to Theorem 3.2, or as in Theorem 3.1), there is a function $\bar{u}(t)$ or $u^{**}(t)$. However in Theorem 3.2, with $n = m + k + 2r$, we have not defined a $u^{**}(t)$, although it obviously can be defined by adding appropriate boundary conditions at the endpoints a and/or b, while maintaining Postulate I. This is due to the non-uniqueness of $u^*(t)$ in the theorem. Nonetheless we shall have use for such a function in Section 4, and introduce it below in the following corollary.

COROLLARY 3.2. *In Theorem* 3.2, *let $f(t)$ be a polynomial $u(t)$, and let*

$$f = u \in \mathscr{U}(\mathscr{A}_k \cap \mathscr{B}_m).$$

Assume

(i) $n = m + k + 2r + 1$ *and $\mathscr{U}(\mathscr{A}_k \cap \mathscr{B}_m)$ satisfies Property* J.

Then, there exists a unique representation $f(t) = u(t) = \bar{u}(t) + \underline{u}(t)$ (see Remark 3.2, *where the zeros of $\bar{u}(t)$ and $\underline{u}(t)$ in (a, b) strictly interlace).*

(ii) $n = m + k + 2r$, $r > 0$, *and $\mathscr{U}(\mathscr{A}_k \cap \mathscr{B}_m)$ satisfies Property* J.

Then there exists a unique representation $f(t) = u(t) = u^*(t) + u^{**}(t)$, *where* $u^{**}(t)$ *is the unique polynomial in* $\mathscr{U}(\mathscr{A}'_{k+1} \cap \mathscr{B}''_{m+1}) = \mathscr{U}(\mathscr{A}''_{k+1} \cap \mathscr{B}'_{m+1})$ *with respect to* $f(t)$ *(as in Theorem 4). The zeros of* $u^{**}(t)$ *strictly interlace the zeros of* $u^*(t)$ *in* (a,b).

(iii) $n = m + k + 2r$, $r > 0$, *and* $\mathscr{U}(\mathscr{A}_k \cap \mathscr{B}_m)$ *does not satisfy Property* J.

Then there exist representations $f(t) = u(t) = u^*(t) + u^{**}(t)$, *where* $u^*(t)$ *is as above, and* $u^{**}(t) \in \mathscr{U}(\mathscr{A}'_{k+1} \cap \mathscr{B}_m) = \mathscr{U}(\mathscr{A}_k \cap \mathscr{B}'_{m+1})$, $f(t) \geqq u^{**}(t) \geqq 0$, *for* $t \in [a,b]$, $Z_{(a,b)}(u^{**}(t)) = 2(r-1)$ *and the zeros of* $u^{**}(t)$ *strictly interlace the zeros of* $u^*(t)$.

PROOF. Similar to the proof of Corollary 3.1. Note that $u(t) \notin \mathscr{U}(\mathscr{A}'_{k+1} \cap \mathscr{B}_m)$, $\mathscr{U}(\mathscr{A}_k \cap \mathscr{B}'_{m+1})$ by the conditions of Theorem 3.2. Q.E.D.

There yet remains the case where $n = m + k + 2r + 1$, and $\mathscr{U}(\mathscr{A}_k \cap \mathscr{B}_m)$ does not satisfy Property J. We prove the following interesting proposition which shows this class to be empty.

PROPOSITION 3.1. *If* $n = m + k + 2r + 1$, $\mathscr{U}(\mathscr{A}_k \cap \mathscr{B}_m)$ *satisfies Postulate* I, *and Property* J *does not hold, then there does* not *exist a* $u \in \mathscr{U}(\mathscr{A}_k \cap \mathscr{B}_m)$ *for which* $u \notin \mathscr{U}(\mathscr{A}'_{k+1} \cap \mathscr{B}_m)$ *and* $u(t) > 0$, *for all* $t \in (a,b)$.

PROOF. Assume the converse. Then, by Theorem 3.2, there exists a

$$u^* \in \mathscr{U}(\mathscr{A}'_{k+1} \cap \mathscr{B}_m) = \mathscr{U}(\mathscr{A}_k \cap \mathscr{B}'_{m+1})$$

satisfying the conditions of Theorem 3.2. Thus $(u-u^*)(t) \in \mathscr{U}(\mathscr{A}_k \cap \mathscr{B}_m)$ and since $u^*(t)$ has an additional zero at both a and b, $Z_{(a,b)}(u - u^*)(t) = 2(r+1)$.

But $k + m + 2(r+1) = n + 1$, and hence $u(t) \equiv u^*(t)$. This is obviously impossible by the definition of $u(t)$ and $u^*(t)$. Accordingly, such a $u(t)$ as described above does not exist. Q.E.D.

Theorem 3.2 and the analysis therein allows us to present the following theorems.

THEOREM 3.3. *Let* f *and* g *be two continuous functions on* $[a,b]$ *such that there exists a polynomial* $v(t) \in \mathscr{U}(\mathscr{A}_k \cap \mathscr{B}_m)$ *being strictly between* f *and* g, *that is*, $f(t) > v(t) > g(t)$ *for all* $t \in [a,b]$. *Then*

(i) *there exists a polynomial* $\underline{u}(t)$ *such that*

(a) $f(t) \geqq \underline{u}(t) \geqq g(t)$, $t \in [a,b]$, *and*

(b) $\underline{u} \in \mathscr{U}(\mathscr{A}_k \cap \mathscr{B}_m)$;

(c) *there are* $n-m-k+1$ *points* $a \leqq s_1 < \cdots < s_{n-m-k+1} \leqq b$ *for which*

$$\underline{u}(s_{n-m-k+1-i}) = \begin{cases} f(s_{n-m-k+1-i}), & i \text{ even} \\ g(s_{n-m-k+1-i}), & i \text{ odd.} \end{cases}$$

$\underline{u}(t)$ is unique if any one *of the following holds*:

(1) $\alpha + \beta > 0$,

(2) *If* $\alpha + \beta = 0$, *and* $\underline{u}(b) \neq f(b)$
or $\underline{u}(a) \neq f(a)$, $n - m - k + 1$ *odd*
$\underline{u}(a) \neq g(a)$, $n - m - k + 1$ *even,*

(3) *If* $\alpha + \beta = 0$, *and Property* J *holds for* $\mathscr{U}(\mathscr{A}_k \cap \mathscr{B}_m)$.

(ii) *Let condition* (c) *and* (2) *be replaced by* (c′) *and* (2′) *where f and g are interchanged. Then there exists a polynomial $\bar{u}(t)$ satisfying* (a), (b), (c′) *and it is unique if* (1), (2′) or (3) *holds.*

Proof. The proof uses the method of Theorem 3.2, and Karlin [2]. Q.E.D.

The restriction $f(t) > v(t) > g(t)$ for all $t \in [a, b]$ may be considerably weakened, both at the endpoints a and b, and in (a, b).

Another generalization of Theorem 3.2 is the following.

Theorem 3.4. *Let $f(t) \in C^n[a, b]$ be a non-negative function defined on $[a, b]$ with the property that $Z_{(a,b)}(f) = 2s \leqq 2r$, and let $f(t)$ otherwise satisfy the conditions in Theorem* 3.2. *Then the results of Theorem* 3.2 *hold with* (iv) *replaced by*:

(iv′) *If $t_{i_1}, \cdots, t_{i_{r-s}}$ are the $r-s$ zeros of $u(t)$ of multiplicity two which remain after removing the zeros of $f(t)$ in (a, b), then $f(t) - u(t)$ vanishes at least once more in each of the open intervals between adjacent pairs of distinct t_{i_k}, and at least once more in each of the intervals $[a, t_{i_1})$ and $(t_{i_{r-s}}, b]$.*

Proof. Use the methods of Theorem 3.2, and Karlin [2].

Corollary 3.3. *In Theorem* 3.4, *let $f(t)$ be a polynomial $u(t)$ (thus $f \in C^n[a, b]$) and let $f = u \in \mathscr{U}(\mathscr{A}_k \cap \mathscr{B}_m)$. Then the results of Corollary* 3.2, *and Proposition* 3.1 *generalize in the obvious manner.*

4. Applications of representation theorems to some extremal problems

We are now prepared to formulate certain extremal properties generalizing the Markoff-Bernstein inequalities. To do this, the following result is necessary.

Proposition 4.1. *Let $p(t)$ be a continuous function on $[a, b]$, $q(t) = \sum_{i=0}^{n} a_i u_i(t)$ and $p(t) > q(t)$, for all $t \in [a, b]$.*

Then, there exists a unique *polynomial* $u^*(t)$ *which satisfies the following*:

(i) *if* $n = m + k + 2r + 1$, $r \geqq 0$,

(a) $p(t) \geqq u(t) \geqq q(t)$, $t \in [a, b]$,

(b) $u - q \in \mathscr{U}(\mathscr{A}_k \cap \mathscr{B}_{m+1})$,

(c) $Z_{(a,b)}(u - q) = 2r$, *r distinct points*,

(d) $p(t) - u(t)$ *vanishes at least once between adjacent zeros of* $u(t) - q(t)$ *and at least once between the largest zero and the endpoint b, and between the smallest zero and the endpoint a*;

(ii) *if* $n = m + k + 2r + 2$, $r \geqq 0$, (a), (c) *and* (d) *hold as above and condition* (b) *is replaced by*

$$\text{(b}'\text{)} \quad u - q \in \mathscr{U}(\mathscr{A}''_{k+1} \cap \mathscr{B}'_{m+1}).$$

PROOF. Apply the previous results to $p(t) - q(t)$. Q.E.D.

Let the conditions of Proposition 4.1 hold, and consider the class of polynomials

$$\mathscr{P} = \{u : p(t) \geqq u(t) \geqq q(t), \text{ for all } t \in [a, b], \text{ and } u - q \in \mathscr{U}(\mathscr{A}_k \cap \mathscr{B}'_{m+1})\}.$$

Let $\bar{B} = \| \bar{B}_{ij} \|$, $i = 1, \cdots, m + 2$; $j = 0, \cdots, n$, where

$$\bar{B}_{ij} = \begin{cases} B_j & i = 1; j = 0, \cdots, n \\ B'_{i-1\, j} & i = 2, \cdots, m+2; j = 0, 1, \cdots, n \end{cases}$$

and $B' = \| B'_{ij} \|$, $i = 1, \cdots, m + 1$; $j = 0, \cdots, n$, as in Section 1, where both B' and $\tilde{A}$, and $\bar{B}$ and $\tilde{A}$ satisfy Postulate I, (if $r = 0$, drop (i) of Postulate I) with $\varepsilon_{m+2}(\bar{B}) = \varepsilon_{m+1}(B')$.

THEOREM 4.1. *Under the above conditions,*

$$\max_{u \in \mathscr{P}} \sum_{j=0}^{n} B_j D^j u(b)$$

is uniquely achieved by u^* *from Proposition* 4.1

PROOF. *Case* I. $n = m + k + 2r + 1$, $r \geqq 0$.

Let $w \in \mathscr{P}$ and $w \not\equiv u^*$. Then $u^* - w$ has at least $2r - 1$ zeros in (a, b) and exactly that number only if $w(a) = u^*(a) = p(a) > q(a)$. However, in this case $N_a(\mathscr{A}_k, \mathscr{B}_m) = 0$, and therefore $u^* - w \in \mathscr{U}(\mathscr{A}''_{k+1} \cap \mathscr{B}'_{m+1})$. Since $u^* - w$ can have at most $2r$ zeros in (a, b), and $2r - 1$ zeros if $u^* - w \in \mathscr{U}(\mathscr{A}''_{k+1} \cap \mathscr{B}'_{m+1})$, we have $u^*(t) - w(t) > 0$, for $t \in (b - \varepsilon, b)$, some $\varepsilon > 0$.

Let $u^*(t) - w(t) = \sum_{i=0}^{n} c_i u_i(t)$, and let $c_j \neq 0$. From Theorem 2.1, we may obtain

$$\operatorname{sgn} c_j = \operatorname{sgn}\,[u^*(t) - w(t)]\,(-1)^{k+2r+j}, \text{ for } t \in (b-\varepsilon, b) = (-1)^{k+j}.$$

Considering $\sum_{j=0}^{n} B_j D^j(u^*(b) - w(b))$ in place of $u^*(t) - w(t)$ and applying the above analysis,

$$\operatorname{sgn} c_j = \operatorname{sgn}\left[\sum_{j=0}^{n} B_j D^j\,(u^*(b) - w(b))\right](-1)^{k+j}.$$

Hence,

$$\sum_{j=0}^{n} B_j D\, u^*(b) > \sum_{j=0}^{n} B_j D^j w(b).$$

If $w(a) = u^*(a) = p(a) > q(a)$, we have used the fact that if $\bar{B}$ and $\tilde{A}$ satisfy Postulate I with $\varepsilon_{m+2}(\bar{B}) = \varepsilon_{m+1}(B')$, then $\bar{B}$ and A' do the same, as was essentially shown in Section 2.

Case II. $n = m + k + 2r + 2,\ r \geqq 0.$

Let $w \in \mathscr{P}$ and $w \not\equiv u^*$. Then $u^* - w$ has at least $2r$ zeros in (a, b), and has exactly $2r$ zeros only if $u^* - w \in \mathscr{U}(\mathscr{A}''_{k+1} \cap \mathscr{B}'_{m+1})$. Otherwise $u^* - w$ has exactly $2r + 1$ zeros in (a, b) and $u^* - w \in \mathscr{U}(\mathscr{A}_k \cap \mathscr{B}'_{m+1})$. We now apply the analysis of Case I, and the theorem is proved. Q.E.D.

By reasoning similar to the above, it is easily shown that

$$\sum_{j=0}^{n} B_j D^j u^*(b) > \sum_{j=0}^{n} B_j D^j q(b).$$

Theorem 4.1 also holds where we do not demand a zero at b, if

$$q(b) < u^*(b) < p(b).$$

We now prove a theorem corresponding to Theorem 4.1 for the minimum case. For ease of notation, let $B^* = B'$. Let $u^{**}(t)$ be constructed as was $u^*(t)$ above such that

(i) $u^{**} - q \in \mathscr{U}(\mathscr{A}_k \cap \mathscr{B}^{*\prime}_{m+2})$ for $n = m + k + 2r + 2,\ r \geqq 0,$

and

(ii) $u^{**} - q \in \mathscr{U}(\mathscr{A}''_{k+1} \cap \mathscr{B}^{*\prime}_{m+2})$ for $n = m + k + 2r + 3,\ r \geqq 0.$

Let $\bar{\bar{B}} = \|\bar{\bar{B}}_{ij}\|,\ i = 0, \cdots, m+3;\ j = 0, \cdots, n$, where

$$\bar{\bar{B}}_{ij} = \begin{cases} B_j & i = 1;\ j = 0, 1, \cdots, n \\ B'^{*}_{i-1\,j} & i = 2, \cdots, m+3;\ j = 0, \cdots, n \end{cases}$$

and $B'^{*} = \|B'^{*}_{ij}\|,\ i = 1, \cdots, m+2;\ j = 0, \cdots n.$

THEOREM 4.2. *Let* $\{B_j\}$, $j = 0, \cdots, n$, *be as defined for Theorem* 4.1. *Then* $\min_{u \in \mathscr{P}} \sum_{j=0}^{n} B_j D^j u(b)$ *is achieved by* $u^{**}(t)$. *This minimum is unique if* $\bar{\bar{B}}$ *and* $\tilde{A}$ *satisfy Postulate* I *(disregarding part* (i)).

PROOF. *Case* I. $n = m + k + 2r + 2$, $r \geqq 0$.

By definition of u^{**}, $u^{**} - q \in \mathscr{U}(\mathscr{A}_k \cap \mathscr{B}^*_{m+2})$ and $u^{**} - q$ has r distinct zeros in (a, b). For each $w \in \mathscr{P}$, $u^{**} - w$ has at least $2r - 1$ zeros in (a, b), and has exactly $2r - 1$ zeros only if $u^{**}(a) = w(a) = p(a) > q(a)$. As has been shown, this is, in essence, another zero and we will regard it as such. Thus $u^{**} - w$ has at least $2r$ zeros in (a, b). If $(u^{**} - w)(t) < 0$ for $t \in (b - \varepsilon, b)$, some $\varepsilon > 0$, then $u^{**} - w$ must have at least $2r + 1$ zeros in (a, b). Since $u^{**} - w \in \mathscr{U}(\mathscr{A}_k \cap \mathscr{B}^*_{m+1})$, we can now apply the methods of Theorem 4.1 to obtain

$$\sum_{j=0}^{n} B_j D^j u^{**}(b) < \sum_{j=0}^{n} B_j D^j w(b).$$

If $(u^{**} - w)(t) > 0$, $t \in (b - \varepsilon, b)$, then $u^{**} - w \in \mathscr{U}(\mathscr{A}_k \cap \mathscr{B}^{*\prime}_{m+2})$ and $(u^{**} - w)(t)$ has exactly $2r$ zeros in (a, b). Assume that $N_b(\mathscr{A}_k, \mathscr{B}_{m+1}) = \beta$. If with $\bar{B}$ in place of B_i^* the degree of the zero at b is greater than β, then $\bar{\bar{B}}$ and $\tilde{A}$ do not satisfy part (ii) of Postulate I and $\sum_{j=0}^{n} B_j D^j u^{**}(b) = \sum_{j=0}^{n} B_j D^j w(b)$, (that is, the addition of a zero at b, and $\sum_{j=0}^{n} B_j D^j u(b)$ are basically identical).

If, on the other hand, it is β, then $(\bar{B})'$ and $\bar{\bar{B}}$ are identical matrices except for the interchange of the first two rows, and $\tilde{A}$ and $\bar{\bar{B}}$ satisfy Postulate I. But in this case, $\varepsilon_{m+3}(\bar{\bar{B}}) = -\varepsilon_{m+3}(\bar{B}')$. Utilizing the fact that $(u^{**} - w)(t) > 0$, $t \in (b - \varepsilon, b)$, and by the analysis of Theorem 4.1, we obtain

$$\sum_{j=0}^{n} B_j D^j u^{**}(b) < \sum_{j=0}^{n} B_j D^j w(b).$$

Case II. $n = m + k + 2r + 3$, $r \geqq 0$.

This case is essentially the same as Case I except that we must consider $(u^{**} - w)(t)$ near a as well as near b. The proof then follows. Q.E.D.

Theorems analogous to Theorems 4.1 and 4.2 can be constructed at the endpoint a. The statements are left to the reader.

Let $\mathscr{P}' = \{u \colon q(t) \leqq u(t) \leqq p(t)$, for all $t \in (a, b)$, $q(t)$ and $p(t)$ as above, and $u - q \in \mathscr{U}(\mathscr{A}_k \cap \mathscr{B}_m)$, $n = m + k + 2r\}$. Assume $\alpha + \beta > 0$. Let $u^*(t)$ be the uniquely defined function satisfying:

(i) $u \in \mathscr{P}'$

(ii) $Z_{(a,b)}(u) = 2r$, and there are r distinct zeros in (a, b)

(iii) $p - u$ vanishes between each pair of adjacent zeros of $u - q$, and at least once between the largest zero and b, and the smallest zero and a.

Define

$$f_{ij}(t) = \frac{p(t) - q(t)}{(b-t)^i(t-a)^j} \text{ for any } 0 \leqq i \leqq \beta,\ 0 \leqq j \leqq \alpha,$$

and assume $f_{ij}(t)$ is strictly decreasing on (a, t_0) and strictly increasing on (t_0, b), for some $t_0 \in (a, b)$.

The simple cases of the next two theorems may be found in Karlin [3].

THEOREM 4.3. *Let the above assumptions prevail. Then,*

$$\max_{u \in \mathscr{P}'} \max_{t \in [a,b]} \frac{u(t) - q(t)}{(b-t)^i(t-a)^j}$$

is attained by $u^*(t)$, and uniquely so if β, $\alpha > 0$.

PROOF. Note that

$$u(t) = u'(t) = \cdots = u^{(l)}(t) = 0 \Leftrightarrow u(t) = Du(t) = \cdots = D^l u(t) = 0.$$

Thus, $$\lim_{t \downarrow a} \frac{u(t) - q(t)}{(b-t)^i(t-a)^j}, \text{ and } \lim_{t \uparrow b} \frac{u(t) - q(t)}{(b-t)^i(t-a)^j}$$

are well defined for $i \leqq \beta$, $j \leqq \alpha$, and the first term is zero for $j < \alpha$, and the latter zero for $i < \beta$.

Assume β, $\alpha > 0$. Let s_0 and s_1 be the first and last zeros, respectively, of $p(t) - u^*(t)$ in (a,b). $w \in \mathscr{P}'$, $u^*(t) - w(t)$ has $2r$ zeros in (a,b), $u^* - w \in \mathscr{U}(\mathscr{A}_k \cap \mathscr{B}_m)$, and $(u^* - w)(t) > 0$ for $t \in (a, s_0]$, $[s_1, b)$. Therefore

$$\frac{u^*(t) - q(t)}{(b-t)^i(t-a)^j} - \frac{w(t) - q(t)}{(b-t)^i(t-a)^j} = \frac{u^*(t) - w(t)}{(b-t)^i(t-a)^j} > 0$$

for $t \in (a, s_0]$, $[s_r, b)$. By the above, for $i < \beta$, the above terms are zero at b, and similarly at a for $j < \alpha$.

For $i = \beta$, $$\lim_{t \downarrow b} \frac{u^*(t) - w(t)}{(b-t)^i(t-a)^j} = \frac{u^{*(\beta)}(b) - w^{(\beta)}(b)}{\beta!\,(b-a)^j}(-1)^\beta.$$

By Theorem 4.1, this value is positive. The proof follows.

If, say $\beta = 0$, then

$$f_{0,j} = \frac{p(t) - q(t)}{(t-a)^j}$$

and the above proof holds except that if $u^*(b) = p(b)$, uniqueness does not necessarily follow. Q.E.D.

Let $g_j(t) = p(t) - q(t)/(t-a)^j$, $0 \leqq j \leqq \alpha$, $\alpha > 0$, and assume $g_j(t)$ is strictly decreasing on $(a,b]$, $N_a(\mathscr{A}_k, \mathscr{B}_m) = \alpha$.

Let $h_i(t) = p(t) - q(t)/(b-t)^i$, $0 \leqq i \leqq \beta$, $\beta > 0$, and assume $h_i(t)$ is strictly increasing on $[a,b)$, $N_b(\mathscr{A}_k, \mathscr{B}_m) = \beta$.

THEOREM 4.4. *Let the above assumptions hold. Then*

(i)
$$\max_{u \in \mathscr{P}'} \max_{t \in [a,b]} \frac{u(t) - q(t)}{(t-a)^j}$$

is uniquely attained by $u^*(t)$, *where* $g_j(t)$ *satisfies the above.*

(ii)
$$\max_{u \in \mathscr{P}'} \max_{t \in [a,b]} \frac{u(t) - q(t)}{(b-t)^i}$$

is uniquely attained by $u^*(t)$, *where* $h_i(t)$ *satisfies the above.*

PROOF. Similar to the proof of Theorem 4.3.

We state the following theorem whose proof, with little change, may be found in Karlin [3].

THEOREM 4.5. *Let* $\mathscr{P}'$ *be as above, and let* $\mathscr{P}'(t_0) = \{w : w \in \mathscr{P}', t_0 \in [a,b], w(t_0) = u^*(t_0)\}$. *Then*

$$\max_{w \in \mathscr{P}'(t_0)} w'(t_0)\,\varepsilon_{u^*}$$

is uniquely attained by u^* *if* $\varepsilon_{u^*} \neq 0$, *where*

$$\varepsilon_{u^*} = \begin{cases} 1 & \text{if } u^{*\prime}(t_0) > 0 \\ -1 & \text{if } u^{*\prime}(t_0) < 0 \\ 0 & \text{if } u^{*\prime}(t_0) = 0. \end{cases}$$

Let Q denote the class of all non-trivial, non-negative polynomials in $\mathscr{U}(\mathscr{A}_k \cap \mathscr{B}_m)$. Then the following theorem holds.

THEOREM 4.6. *Assume that* L *is a linear functional defined on* Q *such that* $L(u) > 0$ *for* $u \in Q$.

(i) *Let* M *be a sublinear functional defined on* Q, that is, $M(u_1 + u_2) \leqq M(u_1) + M(u_2)$, $u_1, u_2 \in Q$. *Then,* $\sup_{u \in Q} M(u)/L(u)$ *is achieved for polynomials possessing a maximum number of zeros.*

(ii) *Let M be a superlinear functional defined on Q, that is,* $M(u_1 + u_2) \geqq M(u_1) + M(u_2)$, $u_1, u_2 \in Q$. *Then,* $\inf_{u \in Q} M(u)/L(u)$ *is achieved for polynomials possessing a maximum number of zeros.*

REMARK 4.1. Let $\tilde{u} \in \mathscr{U}(\mathscr{A}_k \cap \mathscr{B}_m)$ have zeros of degree α and β at a and b, respectively, and $Z_{(a,b)}(\tilde{u}) = s$. Then $\tilde{u}$ is said to have *a maximum number of zeros* if for any other $u \in \mathscr{U}(\mathscr{A}_k \cap \mathscr{B}_m)$ with zeros of degree α and β at a and b, respectively, for which $Z_{(a,b)}(u) = s$, and with either additional zeros at a or b, or in (a, b), we then have $u \equiv 0$.

In other words, $\tilde{u}$ has the maximum number of zeros in (a, b) given the boundary conditions (including additional zeros) which it satisfies.

PROOF OF THEOREM 4.6. First note that the extremal values are finite and are actually attained. We consider only (i), since the proof of (ii) is totally analogous.

Due to the homogeneity of M and L, it is sufficient to consider those $u \in Q$ obeying the normalization $L(u) = 1$. Therefore, it is sufficient to establish for every polynomial $v \in Q$ satisfying $L(v) = 1$, the existence of *a* polynomial $\tilde{u} \in Q$, with a maximum number of zeros, for which $L(\tilde{u}) = 1$, and $M(\tilde{u}) \geqq M(v)$.

Assume that $L(v) = 1$, and v has less than a maximum number of zeros. Then $v \in \mathscr{U}(\mathscr{A}_{\tilde{k}} \cap \mathscr{B}_{\tilde{m}})$ and $Z_{(a,b)}(v) = s$, $n > \tilde{k} + \tilde{m} + s$, where $u \in \mathscr{U}(\mathscr{A}_{\tilde{k}} \cap \mathscr{B}_{\tilde{m}})$ implies $u \in \mathscr{U}(\mathscr{A}_k \cap \mathscr{B}_m)$, and v and $\mathscr{U}(\mathscr{A}_{\tilde{k}} \cap \mathscr{B}_{\tilde{m}})$ have zeros of the same degree at a and at b.

Case I. If $n = \tilde{k} + \tilde{m} + 2r + 1$, then by Corollary 3.3 (and Proposition 3.1), $\mathscr{U}(\mathscr{A}_{\tilde{k}} \cap \mathscr{B}_{\tilde{m}})$ must satisfy Property J, and there exists unique $\underline{u}(t)$ and $\bar{u}(t)$, for which $\underline{u}$, $\bar{u} \in Q$, $\bar{u}$, $\underline{u}$ have maximum number of zeros and $v(t) = \bar{u}(t) + \underline{u}(t)$. Let $\gamma = L(\bar{u})$ and $\sigma = L(\underline{u})$. Then γ, $\sigma > 0$, and $\gamma + \sigma = 1$ since $L(v) = 1$. Thus $v = \gamma(\bar{u}/\gamma) + \sigma(\underline{u}/\sigma)$, and $M(v) \leqq \max\{M(\bar{u})/\gamma, M(\underline{u})/\sigma\}$.

Case II. If $n = \tilde{k} + \tilde{m} + 2r$, $r > 0$, then we apply Corollary 3.3 as above.

Q.E.D.

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REFERENCES

1. F. R. Gantmacher and M. G. Krein, *Oscillatory Matrices and Kernels and Small Vibrations of Mechanical Systems*, (in Russian), 2nd ed., Moscow, 1950.

2. S. Karlin, *Representation theorems for positive functions*, J. Math. Mech. **12** (1963), 559–618.

3. S. Karlin, *Generalized Bernstein inequalities*, Acta Sci. Math. (Szeged) (1966), pp. 1–15.

4. S. Karlin, *Total Positivity*, Stanford Univ. Press, Stanford, Calif., 1968.

5. S. Karlin, *Total positivity, interpolation by splines, and Green's functions of differential operators*, J. Approximation Theory **4** (1971), 91–112.

6. S. Karlin and J. Karon, *On Hermite-Birkhoff interpolation*, J. Approximation Theory **6** (1972), 90–115.

7. S. Karlin and C. A. Micchelli, *The fundamental theorem of algebra for monosplines satisfying boundary conditions*, Israel J. Math. **11** (1972), 405–451.

8. S. Karlin and L. S. Shapley, *Geometry of moment spaces*, Mem. Amer. Math. Soc. **12** (1953).

9. S. Karlin and W. Studden, *Tchebycheff Systems: with Applications in Analysis and Statistics*, Interscience, New York, 1966.

10. C. A. Micchelli and T. J. Rivlin, *Quadrature formulae and Hermite-Birkhoff interpolation, IBM J. Res.* Develop., RC 3396, June, 1971.

11. M. A. Neumark, *Lineare Differentialoperatoren*, Akademie-Verlag, Berlin, 1960.

12. I. J. Schoenberg, *On Hermite-Birkhoff interpolation*, J. Math. Anal. Appl. **16** (1966), 538–543.

13. G. Szegö, *Orthogonal Polynomials*, Amer. Math. Soc., Vol. XXIII, New York, 1969.

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